Control and Cybernetics

vol. **39** (2010) No. 4

Improved fuzzy feedback linearization and Sinswat-transformation control of inverted pendulum*

by

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Abstract: This paper studies the output tracking and almost disturbance decoupling problem of nonlinear control systems with uncertainties via fuzzy logic control and feedback linearization approach. The main contribution of this study is to construct a controller, under appropriate conditions, such that the resulting closed-loop system enjoys for any initial condition and bounded tracking signal the following characteristics: input-to-state stability with respect to disturbance inputs and almost disturbance decoupling, i.e., the influence of disturbances on the L_2 norm of the output tracking error can be arbitrarily attenuated by increasing some adjustable parameters. The underlying theoretical approaches are the differential geometry approach and the composite Lyapunov approach. One example, which cannot be solved by the approach from the first paper (Marino et al., 1989) on the almost disturbance decoupling problem, is proposed in this paper to exploit the fact that the almost disturbance decoupling and the convergence rate performances are easily achieved by virtue of our approach. In order to demonstrate the practical applicability, the paper takes up the study of an inverted pendulum control system.

Keywords: fuzzy logic control, almost disturbance decoupling, feedback linearization approach, composite Lyapunov approach, inverted pendulum control system.

*Submitted: January 2008; Accepted: May 2010.

1. Introduction

A recent development in nonlinear control design is that of feedback linearization, which transforms the original nonlinear system into an equivalent controllable linear system. Feedback linearization is an approach to nonlinear control design, which has attracted a great deal of research interest in recent years (Slotine and Li, 1991; Yang and Calise, 2007). Moreover, feedback linearization approach has been applied successfully to many real control cases (Yang and Calise, 2007). These include control of hydraulic cylinder system (Hahn et al., 1992, 1994), pharmacogenomics system (Floares, 2005, 2006), continuously stirred tank reactor (Guo, 2006), electromagnetic suspension system (Joo and Seo, 1997), pendulum system (Corless and Leitmann, 1981), spacecraft (Sheen and Bishop, 1994), electrohydraulic servosystem (Alleyne, 1998), car-pole system (Bedrossian, 1992) and bank-to-turn missile system (Lee et al., 1997).

In the past few years, the differential geometry approach (Banks, 1988; Nijmeijer and Van Der Schaft, 1990) proved to be an effective means of analysis and design of nonlinear control systems, as it was in the past for the Laplace transform, complex variable theory and linear algebra in relation to linear control systems. The main concept of this approach is to algebraically transform the nonlinear control system into an equivalent linear system, such that the conventional linear control techniques can be utilized (Isidori, 1989).

For many practical control systems, it is difficult to obtain completely accurate mathematical models. Thus, there are inevitable uncertainties in their models. Therefore, the design of a robust controller that deals with uncertainties of a control system is a significant subject for the design of an excellent control system. In this paper, we present a systematic analysis and a simple design scheme that guarantees the globally asymptotical stability of feedback-controlled uncertain system and achieves output tracking performance for a class of nonlinear control systems with uncertainties.

Fuzzy logic control appears to have attracted a great deal attention in the past two decades. Despite the success, many fundamental issues remain unanswered. Almost disturbance decoupling analysis and systematic design are among the most important issues to be further addressed. The almost disturbance decoupling problem, that is the design of a controller which attenuates the effect of the disturbance on the output terminal to an arbitrary degree of accuracy, was originally developed for linear and nonlinear control systems by Willems (1981) and Marino et al. (1989), respectively. Henceforward, the problem has attracted considerable attention and many significant results have been developed for both linear and nonlinear control systems (see Marino et al., 1989; Weiland and Willems, 1989; Marino and Tomei, 1999; Qian and Lin, 2000). Marino et al. (1989) demonstrated that for nonlinear SISO system the almost disturbance decoupling problem may not be solvable, as the following example shows:

$$\dot{x}_1(t) = x_2 + \theta_1(t)$$
$$\dot{x}_2(t) = x_2^3\theta_2(t) + u$$
$$y = x_1$$

where u, y denote the input and output, respectively, and θ_1, θ_2 are the disturbances. This example, though, can be easily solved via the approach proposed in this paper.

Fuzzy logic control has been applied not only to cement kiln or to subway train, but also to other industrial processes. In terms of inference process there are two main classes of fuzzy inference systems (FIS): the Mamdani-type FIS (Mamdani and Assilian, 1975) and the Takagi-Sugeno-Kang (TSK) type FIS (see Takagi and Sugeno, 1985, and Mendel, 2001). In many decision support applications it is important to guarantee the expressive power, easy formalization and interpretability of Mamdani-type FIS. The Mamdani FIS is more widely used, particularly for decision support applications, mostly because of the intuitive and interpretable nature of the rule base. Its design procedure is as follows. First, representing the nonlinear system as the famous Takagi-Sugeno fuzzy model offers an alternative to the conventional model. The control design is carried out based on an aggregation of linear controllers constructed for each local linear element of the fuzzy model via the parallel distributed compensation scheme (Wang et al., 1996). For the stability analysis of the fuzzy system, a lot of studies are reported (see, e.g., Tanaka and Sugeno, 1990, 1992; Lam et al., 2000; Tanaka et al., 2003, and the references therein). The stability and controller design of the fuzzy system can be mainly discussed by Tanaka-Sugeno's theorem (Tanaka and Sugeno, 1990). However, it is difficult to find the common positive definite matrix P for linear matrix inequality (LMI) problem, even if P is a second order matrix (Kawamoto et al., 1992). Moreover, the stability guarantee of the "directly" used fuzzy control for the desired control system is always a debatable point.

Therefore, we propose the acceptable viewpoint that based on the feedback linearization approach a tracking control is proposed in order to guarantee the almost disturbance decoupling property and the uniform ultimate bounded stability of the tracking error system response within an adjustable global final attractor of the zero state. Once the tracking errors are driven to touch the attractor with the desired radius, the Mamdani fuzzy logic control is immediately applied via human expert's knowledge to improve the convergence rate. To overcome the difficulty of finding the common positive definite matrix P for the fuzzy model approach, we propose a new method to guarantee that the closed-loop systems is stable and the almost disturbance decoupling performance is achieved. The design structure is as follows. First, based on the feedback linearization approach a tracking control is proposed in order to guaranteed the almost disturbance decoupling property and the uniform ultimate bounded stability of the tracking error system response within an adjustable global final attractor of the zero state, i.e., such response enters a neighborhood of zero state in finite time and remains within it thereafter. Once the tracking errors are driven to touch the attractor with the desired radius, the conventional fuzzy logic control is immediately applied via human expert's knowledge to improve the convergence rate.

To show the significant applicability of the approach, this paper also describes a successfully derived tracking controller with almost disturbance decoupling for the famous inverted pendulum control system. Throughout the paper, notation $\|\cdot\|$ denotes the usual Euclidean norm or the corresponding induced matrix norm.

2. Controler design

2.1. Feedback linearization controller design

In this paper, we consider the following nonlinear control system with uncertainties and disturbances:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} f_1(x_1, x_2, \ldots, x_n) \\ f_2(x_1, x_2, \ldots, x_n) \\ \vdots \\ f_n(x_1, x_2, \ldots, x_n) \end{bmatrix} + \begin{bmatrix} g_1(x_1, x_2, \ldots, x_n) \\ g_2(x_1, x_2, \ldots, x_n) \\ \vdots \\ g_n(x_1, x_2), \ldots, x_n \end{bmatrix} u$$

$$+ \begin{bmatrix} \Delta f_1 \\ \Delta f_2 \\ \vdots \\ \Delta f_n \end{bmatrix} + \begin{bmatrix} \Delta g_1 \\ \Delta g_2 \\ \vdots \\ \Delta g_n \end{bmatrix} u + \sum_{i=1}^{p} q_i^* \theta_i \tag{2.1a}$$

$$y(t) = h(x_1, x_2, \ldots, x_n) \tag{2.1b}$$

i.e.,

$$\dot{X} = f(X(t)) + g(X(t))u + \Delta f + \Delta g \cdot u + \sum_{i=1}^{p} q_i^* \theta_i$$

$$y(t) = h(X(t))$$

where $X(t) := [\ x_1(x)\ \ x_2(t)\ \ \ldots\ \ x_n(t)\ ]^T \in \Re^n$ is the state vector, $u \in \Re^1$ is the input, $y \in \Re^1$ is the output, $\theta := [\ \theta_1(t)\ \ \theta_2(t)\ \ \ldots\ \ \theta_n(t)\ ]^T$ is a bounded time-varying disturbances vector and $\Delta f := [\ \Delta f_1\ \ \Delta f_2\ \ \ldots\ \ \Delta f_n\ ] \in \Re^n$, $\Delta g := [\ \Delta g_1\ \ \Delta g_2\ \ \ldots\ \ \Delta g_n\ ] \in \Re^n$ are the system uncertainties; $f, g, q_1^*, \ldots, q_p^*$ are smooth vector fields on $\Re^n$, and $h(X(t)) \in \Re^1$ is a smooth function. The nominal system is then defined as follows:

$$\dot{X} = f(X(t)) + g(X(t))u \tag{2.2a}$$

$$y(t) = h(X(t)) \tag{2.2b}$$

The nominal system (2.2) possesses relative degree r (Henson and Seborg, 1991), i.e., there exists a positive integer $1 \leq r < \infty$

$$L_g L_f^k h(X(t)) = 0, \quad k < r-1 \tag{2.3}$$

$$L_g L_f^{r-1} h(X(t)) \neq 0 \tag{2.4}$$

for all $X \in \Re^n$ and $t \in [0, \infty)$, where the operator L is the Lie derivative (Isidori, 1989). The desired output trajectory $y_d(t)$ and its first r derivatives are all uniformly bounded and

$$\left\| \left[y_d(t), y_d^{(1)}(t), \ldots, y_d^{(r)}(t) \right] \right\| \leq B_d, \tag{2.5}$$

where B_d is some positive constant. For the uncertainties, there exist smooth functions $\delta_1(\cdot), \delta_2(\cdot) : \Re^n \to \Re$ such that the uncertainties Δf and Δg in (2.1) satisfy $\Delta f(X) = g(X)\delta_1(X)$ and $\Delta g(X) = g(X)\delta_2(X)$.

Under the assumption of well-defined relative degree, it has been shown (Isidori, 1989) that the mapping

$$\phi : \Re^n \to \Re^n \tag{2.6}$$

defined as

$$\phi_i(X(t)) := \xi_i(t) = L_f^{i-1} h(X(t)), \quad i = 1, 2, \ldots, r \tag{2.7}$$

$$\phi_k(X(t)) := \eta_k(t), \quad k = r+1, r+2, \ldots, n \tag{2.8}$$

and satisfying

$$L_k \phi_k(X(t)) = 0, \quad k = r+1, r+2, \ldots, n \tag{2.9}$$

is a diffeomorphism onto image. For the sake of convenience, define the trajectory error to be

$$e_i(t) := \xi_i(t) - y_d^{(i-1)}(t), \quad i = 1, 2, \ldots, r \tag{2.10}$$

$$e := [\ e_1(t) \ \ e_2(t) \ \ \ldots \ \ e_r(t)\ ]^T \in \Re^r \tag{2.11}$$

the trajectory error with parameterization

$$\bar{e}_i(t) := \varepsilon^{i-1} e_i(t), \quad i = 1, 2, \ldots, r \tag{2.12}$$

$$\bar{e} := [\ \bar{e}_1(t) \ \ \bar{e}_2(t) \ \ \ldots \ \ \bar{e}_r(t)\ ]^T \in \Re^r \tag{2.13}$$

where ε is some adjustable constant, and

$$\xi(t) := [\ \xi_1(t) \ \ \xi_2(t) \ \ \ldots \xi_r(t)\ ]^T \in \Re^r \tag{2.14a}$$

$$\eta(t) := [\ \eta_{r+1}(t) \ \ \eta_{r+2}(t) \ \ \ldots \eta_n(t)\ ]^T \in \Re^{n-r} \tag{2.14b}$$

$$\begin{aligned} q(\xi(t), \eta(t)) &:= [\ L_f \phi_{r+1}(t) \ \ L_f \phi_{r+2}(t) \ \ \ldots \ \ L_f \phi_n(t)\ ]^T \\ &:= [\ q_{r+1} \ \ q_{r+2} \ \ \ldots \ \ q_n\ ]^T \end{aligned} \tag{2.14c}$$

Define a phase-variable canonical matrix A_c to be

$$A_c := \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0\dots & 0 \\ & \vdots & & & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -\alpha_1 & -\alpha_2 & -\alpha_3 & \dots & -\alpha_r \end{bmatrix}_{r\times r} \tag{2.15}$$

where $\alpha_1, \alpha_2, \dots, \alpha_r$ are any chosen parameters such that A_c is Hurwitz and the matrix B is to be

$$B := [\ 0 \quad 0 \quad \dots \quad 0 \quad 1\ ]^T_{r\times 1} \tag{2.16}$$

To facilitate the forthcoming discussion, a coordinate transformation is introduced here. Since the pair (A_c, B) is controllable, there exist matrices $M \in \Re^{r\times(r-1)}$ and $K \in \Re^{1\times r}$ (Sinswat and Fallside, 1977) such that

$$(A_c + BK)M = MD$$

where $D \in \Re^{(r-1)\times(r-1)}$ is an adjustable diagonal matrix

$$D := \begin{bmatrix} -\lambda_1 & 0 & 0 & \dots & 0 \\ 0 & -\lambda_2 & 0 & 0\dots & 0 \\ & \vdots & & & \vdots \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & -\lambda_{r-1} \end{bmatrix} \tag{2.17}$$

where λ_i, $i = 1, 2, \dots, r-1$ are positive constants and $\lambda_{\min} := \min(\lambda_1, \dots, \lambda_{r-1})$. Based on the procedures of Elghezawi et al. (1983) we can construct the generalized inverses $B^g \in \Re^{1\times r}$, $M^g \in \Re^{(r-1)\times r}$ of matrices B and M such that

$$B^g B = 1, \quad B^g M = 0, \quad M^g M = I_{(r-1)\times(r-1)}, \quad M^g B = 0. \tag{2.18}$$

Define the associated tracking error $\tilde{e}$ as

$$\tilde{e} := \begin{bmatrix} \tilde{e}_1 \\ \tilde{e}_2 \end{bmatrix} = \begin{bmatrix} M^g \\ B^g \end{bmatrix} \bar{e} := W\bar{e}, \quad W := \begin{bmatrix} M^g \\ B^g \end{bmatrix} \tag{2.19a}$$

where W is invertible with

$$W^{-1} = [MB]_{r\times r} \tag{2.19b}$$

and

$$\|M^g \bar{e}\| \geq \|\bar{e}_1\|. \tag{2.19c}$$

ASSUMPTION 1 *For all* $t \geq 0$, $\eta \in \Re^{n-r}$ *and* $\xi \in \Re^{r}$, *there exists a positive constant* L *such that the following inequality holds:*

$$\|q_{22}(\eta,\tilde{e}) - q_{22}(\eta,0)\| \leq L(\|\tilde{e}_1\| + \|\tilde{e}_2\|) \tag{2.20}$$

where $q_{22}(\eta,\tilde{e}) := q(\xi,\eta)$.

ASSUMPTION 2 *There exist known functions* $\beta_1(\cdot), \beta_2(\cdot) : \Re^n \to \Re^+$ *such that*

$$1 + \delta_2(X) \geq \beta_1 \tag{2.21a}$$

$$\left\|\delta_2 y_d^{(r)} - (1+2\delta_2)\varepsilon^{-r}\bar{\bar{e}} + \delta_1 d - \delta_2 c + B^g \bar{e}\right\| \leq \beta_2\|\tilde{e}_2\| \tag{2.21b}$$

where

$$d := L_g L_f^{r-1} h(X(t)) \tag{2.22a}$$

$$c := L_f^r h(X(t)) \tag{2.22b}$$

$$\bar{\bar{e}} = \alpha_1\bar{e}_1 + \alpha_2\bar{e}_2 + \ldots + \alpha_r\bar{e}_r. \tag{2.22c}$$

DEFINITION 1 *(Khalil, 1996) A continuous function* $\alpha : [0,a) \to [0,\infty)$ *is said to belong to class K if it is strictly increasing and* $\alpha(0) = 0$.

DEFINITION 2 *(Khalil, 1996) A continuous function* $\beta : [0,a) \times [0,\infty) \to [0,\infty)$ *is said to belong to class KL if, for each fixed s, the mapping* $\beta(r,s)$ *belongs to class K with respect to r and, for each fixed r, the mapping* $\beta(r,s)$ *is decreasing with respect to s and* $\beta(r,s) \to 0$ *as* $s \to \infty$.

DEFINITION 3 *(Khalil, 1996) Consider a system* $\dot{x} = f(t,x,\theta)$, *where* $f : [0,\infty) \times \Re^n \times \Re^n \to \Re^n$ *is piecewise continuous in t and locally Lipschitz in x and θ. This system is said to be input-to-state stable if there exist a class KL function β, a class K function γ and positive constants* k_1 *and* k_2 *such that for any initial state* $x(t_0)$ *with* $\|x(t_0)\| < k_1$ *and any bounded input* $\theta(t)$ *with* $\sup_{t \geq t_0}\|\theta(t)\| < k_2$, *the state exists and satisfies*

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0) + \gamma\left(\sup_{t_0 \leq \tau \leq t}\|\theta(\tau)\|\right) \tag{2.23a}$$

for all $t \geq t_0 \geq 0$.

Now we formulate the tracking problem with almost disturbance decoupling as follows:

DEFINITION 4 *(Marino and Tomei, 1999) The tracking problem with almost disturbance decoupling is said to be globally solvable by the state feedback controller u for the transformed-error system by a global diffeomorphism* (2.6) *and coordinate transformation* (2.18), *if the controller u enjoys the following properties:*

i) It is input-to-state stable with respect to disturbance inputs.

ii) For any initial value $\tilde{x}_{e0} := \begin{bmatrix} \tilde{e}_1(t_0) & \tilde{e}_2(t_0) & \eta(t_0) \end{bmatrix}^T$, *for any* $t \geq t_0$ *and for any* $t_0 \geq 0$

$$|y(t) - y_d(t)| \leq \beta_{11}(\|x(t_0)\|, t - t_0) + \frac{1}{\sqrt{\beta_{22}}}\beta_{33}\left(\sup_{t_0 \leq \tau \leq t} \|\theta(\tau)\|\right) \quad (2.23b)$$

and

$$\int_{t_0}^{t} [y(\tau) - y_d(\tau)]^2 \, d\tau \leq \frac{1}{\beta_{44}}\left[\beta_{55}\left(\|\tilde{x}_{e0}\|\right) + \int_{t_0}^{t} \beta_{33}\left(\|\theta\left(\tau\right)\|^2\right) d\tau\right] \quad (2.23c)$$

where β_{22}, β_{44} *are positive constants,* β_{33}, β_{55} *are class K functions and* β_{11} *is a class KL function.*

Definition 5 *Consider the following dynamical system*

$$\dot{z}(t) = f(t, z(t)), \quad z \in \Re^p, \; z(t_0) := z_0$$

where $z \in \Re^p$ *is the state and* $f(\cdot)$ *is a smooth function. We use* $z(t; t_0, z_0)$ *to denote the solution of system with* $z(t_0; t_0, z_0) = z_0$. *A closed set* S *is called a global final attractor for the trajectories* $z(\cdot) : [t, \infty) \to \Re^p$, $z(t_0) = z_0$ *of the system, if for any initial state* z_0, *there exists a finite constant* $T(z_0, S) \in [0, \infty)$ *such that*

$$z(t_0; t_0, z_0) \in S, \; \forall t \geq t_0 + T(z_0, S).$$

Now we present our main result.

Theorem 1 *Suppose that there exists a continuously differentiable function* $V_0\Re^{n-r} \to \Re^+$ *such that the following three inequalities hold for all* $\eta \in \Re^{n-r}$:

$$\text{(a)} \quad k_1\|\eta\|^2 \leq V_0(\eta) \leq k_2\|\eta\|^2, \quad k_1, k_2 > 0 \quad (2.24a)$$

$$\text{(b)} \quad (\Delta_\eta V_0)^T q_{22}(\eta, 0) \leq k_3\|\eta\|^2, \quad k_3 > 0 \quad (2.24b)$$

$$\text{(c)} \quad \|\Delta_\eta V_0\| \leq k_4\|\eta\|, \quad k_4 > 0, \quad (2.24c)$$

then the tracking problem with almost disturbance decoupling is globally solvable by the controller defined by

$$u_{feedback} = \left[L_g L_f^{r-1} h(X(t))\right]^{-1} \Big\{ - L_f^r h(X) + y_d^{(r)} - 2\varepsilon^{-r}\alpha_1 \left[L_f^0 h(X) - y_d\right]$$
$$-2\varepsilon^{1-r}\alpha_2 \left[L_f^1 h(X) - y_d^{(1)}\right] - \ldots - 2\varepsilon^{-1}\alpha_r \left[L_f^{r-1} h(X) - y_d^{(r-1)}\right] - mB^g\bar{e}\Big\} \quad (2.25)$$

where m *is an adjustable positive constant and the influence of disturbances on the* L_2 *norm of the tracking error can be arbitrarily attenuated by increasing the* *following adjustable parameter $N_2 > 1$:*

$$N_2 = \min\{k_{11}, k_{22}, k_{33}\} \tag{2.26a}$$

$$k_{11} := \frac{\alpha\lambda_{\min}}{\varepsilon} - \frac{1}{2} - \left(\frac{\alpha}{\varepsilon}\|M^g\phi_\xi\|\right)^2 \tag{2.26b}$$

$$k_{22} := -\left(\frac{\alpha}{\varepsilon}\|M^g A_c B\| + \frac{1}{\varepsilon}\|B^g A_c M\|\right)^2 - \frac{1}{\varepsilon}\|B^g A_c B\| \\ +\varepsilon^{r-1} + \varepsilon^{r-1} m\beta_1 - \frac{1}{16} - \left(\frac{1}{\varepsilon}\|B^g\phi_\xi\|\right)^2 - \beta_2\varepsilon^{r-1} \tag{2.26c}$$

$$k_{33} := \mu k_3 - 5(\mu L k_4)^2 - (\mu L k_4\|\phi_\eta\|)^2 \tag{2.26d}$$

$$\phi_\xi := \begin{bmatrix} \varepsilon\dfrac{\partial}{\partial X}hq_1^* & \cdots & \varepsilon\dfrac{\partial}{\partial X}hq_p^* \\ \vdots & & \vdots \\ \varepsilon^r\dfrac{\partial}{\partial X}L_f^{r-1}hq_1^* & \cdots & \varepsilon^r\dfrac{\partial}{\partial X}L_f^{r-1}hq_q^* \end{bmatrix} \tag{2.26e}$$

$$\phi_\eta := \begin{bmatrix} \dfrac{\partial}{\partial X}\phi_{r+1}q_1^* & \cdots & \dfrac{\partial}{\partial X}\phi_{r+1}q_p^* \\ \vdots & & \vdots \\ \dfrac{\partial}{\partial X}\phi_n q_1^* & \cdots & \dfrac{\partial}{\partial X}\phi_n q_q^* \end{bmatrix} \tag{2.26f}$$

$$N_1 := \frac{3}{4}\left(\sup_{t_0\le\tau\le t}\|\theta(\tau)\|\right)^2 \tag{2.26g}$$

where $\alpha \ge 2$ and μ are strictly positive constants to be adjusted. Moreover, the output tracking error of system (2.1) is exponentially attracted into a sphere $B_{\underline{r}}$, $\underline{r} = \sqrt{\frac{N_1}{N_2}}$, with an exponential rate of convergence

$$\frac{1}{2}\left(\frac{N_2}{\omega_2} - \frac{N_1}{\omega_2\underline{r}^2}\right) := \frac{1}{2}\alpha^*. \tag{2.26h}$$

Proof. Applying the co-ordinate transformation (2.6) yields

$$\begin{aligned}\dot{\xi}_1(t) = \frac{\partial\phi_1}{\partial X}\frac{dX}{dt} &= \frac{\partial h(X(t))}{\partial X}\left[f + g\cdot u + \Delta f + \Delta g\cdot u + \sum_{i=1}^{p} q_i^*\theta_i\right] \\ &= L_f^1 h(X(t)) + L_g L_f^0 h(X(t))u + \frac{\partial h(X)}{\partial X}(\Delta f + \Delta g\cdot u) + \frac{\partial h(X)}{\partial X}\sum_{i=1}^{p} q_i^*\theta_i \\ &= L_f^1 h(X(t)) + \frac{\partial h(X)}{\partial X}(\Delta f + \Delta g\cdot u) + \frac{\partial h(X)}{\partial X}\sum_{i=1}^{p} q_i^*\theta_i\end{aligned}$$

$$
\begin{aligned}
&= \xi_2(t) + \frac{\partial h}{\partial X} g(X)\delta_1(X) + \frac{\partial h}{\partial X} g(X)\delta_2(X)u + \frac{\partial h(X)}{\partial X}\sum_{i=1}^{p} q_i^*\theta_i \\
&= \xi_2(t) + \sum_{i=1}^{p} \frac{\partial h(X)}{\partial X} q_i^*\theta_i
\end{aligned} \tag{2.27}
$$

$$
\begin{aligned}
\dot{\xi}_{r-1}(t) &= \frac{\partial \phi_{r-1}}{\partial X}\frac{dX}{dt} = \frac{\partial L_f^{r-2}h(X(t))}{\partial X}\left[f + g\cdot u + \Delta f + \Delta g\cdot u + \sum_{i=1}^{p} q_i^*\theta_i\right] \\
&= L_f^{r-1}h(X(t)) + L_g L_f^{r-2}h(X(t))u + \frac{\partial L_f^{r-2}h(X(t))}{\partial X}(\Delta f + \Delta g\cdot u) \\
&\quad + \frac{\partial L_f^{r-2}h(X(t))}{\partial X}\sum_{i=1}^{p} q_i^*\theta_i \\
&= L_f^{r-1}h(X(t)) + \frac{\partial L_f^{r-2}h(X(t))}{\partial X}(\Delta f + \Delta g\cdot u) + \frac{\partial L_f^{r-2}h(X(t))}{\partial X}\sum_{i=1}^{p} q_i^*\theta_i \\
&= \xi_r(t) + \frac{\partial}{\partial X}L_f^{r-2}h(X)g(X)\delta_1(X) + \frac{\partial}{\partial X}L_f^{r-2}h(X)g(X)\delta_2(X)u \\
&\quad + \frac{\partial L_f^{r-2}h(X(t))}{\partial X}\sum_{i=1}^{p} q_i^*\theta_i \\
&= \xi_r(t) + \sum_{i=1}^{p} \frac{\partial L_f^{r-2}h(X(t))}{\partial X} q_i^*\theta_i
\end{aligned} \tag{2.28}
$$

$$
\begin{aligned}
\dot{\xi}_{r}(t) &= \frac{\partial \phi_{r}}{\partial X}\frac{dX}{dt} = \frac{\partial L_f^{r-1}h(X(t))}{\partial X}\left[f + g\cdot u + \Delta f + \Delta g\cdot u + \sum_{i=1}^{p} q_i^*\theta_i\right] \\
&= L_f^{r}h(X) + L_g L_f^{r-1}h(X(t))u + \frac{\partial L_f^{r-1}h(X(t))}{\partial X}(\Delta f + \Delta g\cdot u) \\
&\quad + \frac{\partial L_f^{r-1}h(X(t))}{\partial X}\sum_{i=1}^{p} q_i^*\theta_i \\
&= L_f^{r}h(X) + L_g L_f^{r-1}h(X)u + \frac{\partial}{\partial X}L_f^{r-1}h(X)g(X)\delta_1(X) \\
&\quad + \frac{\partial}{\partial X}L_f^{r-1}h(X)g(X)\delta_2(X)u + \frac{\partial L_f^{r-1}h(X(t))}{\partial X}\sum_{i=1}^{p} q_i^*\theta_i \\
&= L_f^{r}h(X) + L_g L_f^{r-1}h(X)\left[(1+\delta_2(X))u + \delta_1(X)\right] \\
&\quad + \sum_{i=1}^{p} \frac{\partial L_f^{r-1}h(X(t))}{\partial X} q_i^*\theta_i
\end{aligned} \tag{2.29}
$$

$$\begin{aligned}
\dot{\eta}_k &= \frac{\partial \phi_k(X)}{\partial X}\frac{dX}{dt} = \frac{\partial \phi_k(X)}{\partial X}\left[f + g\cdot u + \Delta f + \Delta g \cdot u + \sum_{i=1}^{p} q_i^* \theta_i\right] \\
&= \frac{\partial \phi_k(X)}{\partial X} f + \frac{\partial \phi_k(X)}{\partial X} gu + \frac{\partial \phi_k(X)}{\partial X}(\Delta f + \Delta g \cdot u) + \frac{\partial \phi_k(X)}{\partial X}\sum_{i=1}^{p} q_i^* \theta_i \\
&= L_f \phi_k(X) + \frac{\partial \phi_k(X)}{\partial X}(\Delta f + \Delta g \cdot u) + \frac{\partial \phi_k(X)}{\partial X}\sum_{i=1}^{p} q_i^* \theta_i \\
&= L_f \phi_k + \frac{\partial \phi_k}{\partial X} g(X)\delta_1(X) + \frac{\partial \phi_k}{\partial X} g(X)\delta_2(X)u + \frac{\partial \phi_k(X)}{\partial X}\sum_{i=1}^{p} q_i^* \theta_i \\
&= L_f \phi_k + \sum_{i=1}^{p} \frac{\partial \phi_k(X)}{\partial X} q_i^* \theta_i \qquad (2.30)
\end{aligned}$$

$$k = r+1, r+2, \ldots, n$$

Since

$$c(\xi(t)\eta(n)) := L_f^r h(X(t)) \qquad (2.31)$$

$$d(\xi(t)\eta(t)) := L_g L_f^{r-1} h(X(t)) \qquad (2.32)$$

$$q_k(\xi(t), \eta(t)) = L_f \phi_k(X), \quad k = r+1, r+2, \ldots, n \qquad (2.33)$$

the dynamic equations of system (2.1) in the new co-ordinates appear as follows:

$$\dot{\xi}_i(t) = \xi_{i+1}(t) + \sum_{i=1}^{p} \frac{\partial}{\partial X} L_f^{i-1} h q_i^* \theta_i, \quad i = 1, 2, \ldots, r-1 \qquad (2.34)$$

$$\begin{aligned}
\dot{\xi}_r(t) &= c(\xi(t)\eta(n)) + d(\xi(t)\eta(t))\left[(1+\delta_2(X))u + \delta_1(X)\right] \\
&\quad + \sum_{i=1}^{p} \frac{\partial}{\partial X} L_f^{i-1} h q_i^* \theta_i, \qquad (2.35)
\end{aligned}$$

$$\dot{\eta}_k(t) = q_k(\xi(t), \eta(t)) + \sum_{i=1}^{p} \frac{\partial}{\partial X}\phi_k(X) q_i^* \theta_i, \quad k = r+1, \ldots, n \qquad (2.36)$$

$$y(t) = \xi_1(t). \qquad (2.37)$$

Define

$$\begin{aligned}
v &:= y_d^{(r)} - 2\varepsilon^{-r}\alpha_1 \left\lfloor L_f^0 h(X) - y_d \right\rfloor - 2\varepsilon^{1-r}\alpha_2 \left\lfloor L_f^1 h(X) - y_d^{(1)} \right\rfloor \\
&\quad - \ldots - 2\varepsilon^{-1}\alpha_r \left\lfloor L_f^{r-1} h(X) - y_d^{(r-1)} \right\rfloor - mB^g \bar{e} \qquad (2.38)
\end{aligned}$$

According to equations (2.7), (2.10), (2.31) and (2.32), the tracking controller can be rewritten as

$$u = d^{-1}[-c + v] \qquad (2.39)$$

By substituting equation (2.39) into (2.35), the dynamic equations of system (2.1) can be written as follows:

$$\begin{bmatrix} \dot{\xi}_1(t) \\ \dot{\xi}_2(t) \\ \vdots \\ \dot{\xi}_{r-1}(t) \\ \dot{\xi}_r(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0\dots & 0 \\ & \vdots & & & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} \dot{\xi}_1(t) \\ \dot{\xi}_2(t) \\ \vdots \\ \dot{\xi}_{r-1}(t) \\ \dot{\xi}_r(t) \end{bmatrix}$$
$$+ \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \{\delta_1 d - \delta_2 c + (1+\delta_2)v\} + \begin{bmatrix} +\sum_{i=1}^{p} \dfrac{\partial}{\partial X} h q_i^* \theta_i \\ +\sum_{i=1}^{p} \dfrac{\partial}{\partial X} L_f^1 h q_i^* \theta_i \\ \vdots \\ +\sum_{i=1}^{p} \dfrac{\partial}{\partial X} L_f^{r-1} h q_i^* \theta_i \end{bmatrix} \tag{2.40}$$

$$\begin{bmatrix} \dot{\eta}_{r+1}(t) \\ \dot{\eta}_{r+2}(t) \\ \vdots \\ \dot{\eta}_{n-1}(t) \\ \dot{\eta}_n(t) \end{bmatrix} = \begin{bmatrix} q_{r+1}(t) \\ q_{r+2}(t) \\ \vdots \\ q_{n-1}(t) \\ q_n(t) \end{bmatrix} + \begin{bmatrix} \sum_{i=1}^{p} \dfrac{\partial}{\partial X} \phi_{r+1} q_i^* \theta_i \\ \sum_{i=1}^{p} \dfrac{\partial}{\partial X} \phi_{r+2} q_i^* \theta_i \\ \vdots \\ \sum_{i=1}^{p} \dfrac{\partial}{\partial X} \phi_{n-1} q_i^* \theta_i \\ \sum_{i=1}^{p} \dfrac{\partial}{\partial X} \phi_n q_i^* \theta_i \end{bmatrix} \tag{2.41}$$

$$y = \begin{bmatrix} 1 & 0 & \cdots & 0 & 1 \end{bmatrix}_{1\times r} \begin{bmatrix} \xi_1(t) \\ \xi_2(t) \\ \vdots \\ \xi_{r-1}(t) \\ \xi_r(t) \end{bmatrix}_{r\times 1} = \xi_1(t) \tag{2.42}$$

Upon combining equations (2.10), (2.12), (2.15) and (2.38), it can be easily verified that equations (2.40)-(2.42) can be transformed into the following form:

$$\dot{\eta}(t) = q(\xi(t), \eta(t)) + \phi_\eta \theta := q_{22}(\eta(t), \tilde{e}) + \phi_\eta \theta \tag{2.43a}$$

$$\varepsilon\dot{\bar{e}}(t) = A_c\bar{e} + B\varepsilon^r\left\{\delta_1 d - \delta_2 c + \delta_2 y_d^{(r)} - (1+2\delta_2)\varepsilon^{-r}\bar{\bar{e}} - m(1+\delta_2)B^g\bar{e}\right\} \\ +\phi_\xi\theta \tag{2.43b}$$

$$y(t) = \xi_1(t). \tag{2.44}$$

Under the coordinate transformation (2.19), the subsystem (2.43b) becomes

$$\begin{bmatrix} \dot{\tilde{e}}_1 \\ \dot{\tilde{e}}_2 \end{bmatrix} = \frac{1}{\varepsilon}\begin{bmatrix} M^g A_c M & M^g A_c B \\ B^g A_c M & B^g A_c B \end{bmatrix}\begin{bmatrix} \tilde{e}_1 \\ \tilde{e}_2 \end{bmatrix} + \begin{bmatrix} M^g\phi_\xi\theta \\ B^g\phi_\xi\theta \end{bmatrix} \\ + \begin{bmatrix} 0_{(r-1)\times 1} \\ 1 \end{bmatrix}\left[\varepsilon^{r-1}\left(\delta_2 y_d^{(r)} - (1+2\delta_2)\varepsilon^{-r}\bar{\bar{e}} + \delta_1 d - \delta_2 c - m(1+\delta_2)B^g\bar{e}\right)\right] \tag{2.45}$$

We consider $V(\tilde{e},\eta)$, defined by a weighted sum of $V_0(\eta)$ and $V_1(\tilde{e})$,

$$V(\tilde{e},\eta) := V_1(\tilde{e}) + \mu V_0(\eta) \tag{2.46}$$

as a composite Lyapunov function of the system (2.43a) and (2.45) (Khorasani and Kokotovic, 1986; Marino and Kokotovic, 1988), where $V_1(\tilde{(e)})$ satisfies

$$V_1(\tilde{e}) := \frac{1}{2}(\alpha\|\tilde{e}_1\|^2 + \|\tilde{e}_2\|^2) \tag{2.47}$$

In view of (2.20)-(2.22), (2.24) and (2.25), the derivative of $V(\tilde{e},\eta)$ along the trajectories of (2.43a) and (2.45) is given by

$$\begin{aligned}
\dot{V} = \dot{V}_1 + \mu\dot{V}_0 &= \frac{\alpha}{2}\left(\tilde{e}_1^T\dot{\tilde{e}}_1 + \dot{\tilde{e}}_1^T\tilde{e}_1\right) + \frac{1}{2}\left(\tilde{e}_2^T\dot{\tilde{e}}_2 + \dot{\tilde{e}}_2^T\tilde{e}_2\right) + \mu\left(\frac{\partial V_0}{\partial\eta}\right)^T\dot{\eta} \\
&= \frac{\alpha}{2\varepsilon}\big[\tilde{e}_1^T(M^g A_c M\tilde{e}_1 + M^g A_c B\tilde{e}_2 + M^g\phi_\xi\theta) \\
&\quad +(M^g A_c M\tilde{e}_1 + M^g A_c B\tilde{e}_2 + M^g\phi_\xi\theta)^T\tilde{e}_1\big] + \frac{1}{2\varepsilon}\Big\{\tilde{e}_2^T\big[B^g A_c M\tilde{e}_1 + B^g A_c B\tilde{e}_2 \\
&\quad +\varepsilon^r(\delta_2 y_d^{(r)} - (1+2\delta_2)\varepsilon^{-r}\bar{\bar{e}} + \delta_1 d - \delta_2 c - m(1+\delta_2)B^g\bar{e}) + B^g\phi_\xi\theta\big]\Big\} \\
&\quad +\Big[B^g A_c M\tilde{e}_1 + B^g A_c B\tilde{e}_2 + \varepsilon^r(\delta_2 y_d^{(r)} - (1+2\delta_2)\varepsilon^{-r}\bar{\bar{e}} + \delta_1 d - \delta_2 c \\
&\quad -m(1+\delta_2)B^g\bar{e}) + B^g\phi_\xi\theta\Big]^T\tilde{e}_2 + \mu\left(\frac{\partial V_0}{\partial\eta}\right)^T q(\xi,\eta) + \phi_\eta\theta \\
&= \frac{\alpha}{\varepsilon}\tilde{e}_1^T(M^g A_c M\tilde{e}_1 + M^g A_c B\tilde{e}_2) + \frac{\alpha}{\varepsilon}\tilde{e}_1^T M^g\phi_\xi\theta + \frac{\tilde{e}_2^T}{\varepsilon}\big[B^g A_c M\tilde{e}_1 \\
&\quad +B^g A_c B\tilde{e}_2 + \varepsilon^r(\delta_2 y_d^{(r)} - (1+2\delta_2)\varepsilon^{-r}\bar{\bar{e}} + \delta_1 d - \delta_2 c - m(1+\delta_2)B^g\bar{e})\big] \\
&\quad +\frac{\tilde{e}_2^T}{\varepsilon}B^g\phi_\xi\theta + \mu\left(\frac{\partial V_0}{\partial\eta}\right)^T[q_{22}(\eta,\tilde{e}) + \phi_\eta\theta - q_{22}(\eta,0) + q_{22}(\eta,0)]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\alpha}{\varepsilon}\tilde{e}_1^T M^g A_c M \tilde{e}_1 + \frac{\alpha}{\varepsilon}\tilde{e}_1^T M^g A_c B \tilde{e}_2 + \frac{\alpha}{\varepsilon}\tilde{e}_1^T M^g \phi_\xi \theta + \frac{\tilde{e}_2^T}{\varepsilon} B^g A_c M \tilde{e}_1 \\
&+ \frac{\tilde{e}_2^T}{\varepsilon} B^g A_c B \tilde{e}_2 + \tilde{e}_2^T \varepsilon^{r-1} (\delta_2 y_d^{(r)} - (1+2\delta_2)\varepsilon^{-r}\overline{\overline{e}} + \delta_1 d - \delta_2 c \\
&- m(1+\delta_2) B^g \overline{e} + B^g \overline{e} - B^g \overline{e}) + \frac{\tilde{e}_2^T}{\varepsilon} B^g \phi_\xi \theta \\
&+ \mu \left(\frac{\partial V_0}{\partial \eta}\right)^T [q_{22}(\eta, \tilde{e}) + \phi_\eta \theta - q_{22}(\eta, 0) + q_{22}(\eta, 0)] \\
&\leq \frac{\alpha}{\varepsilon}\tilde{e}_1^T D \tilde{e}_1 + \frac{1}{\varepsilon}\left(\alpha \|M^g A_c B\| + \|B^g A_c M\|\right) \|\tilde{e}_1\| \|\tilde{e}_2\| + \frac{1}{\varepsilon}\|\tilde{e}_2\|^2 \|B^g A_c B\| \\
&+ \varepsilon^{r-1} \|\delta_2 y_d^{(r)} - (1+2\delta_2)\varepsilon^{-r}\overline{\overline{e}} + \delta_1 d - \delta_2 c + B^g \overline{e}\| \|\tilde{e}_2\| - \tilde{e}_2^T \varepsilon^{r-1} B^g \overline{e} \\
&- \tilde{e}_2^T \varepsilon^{r-1} m(1+\delta_2) B^g \overline{e} + \mu \left(\frac{\partial V_0}{\partial \eta}\right)^T [q_{22}(\eta, \tilde{e} + \phi_\eta \theta - q_{22}(\eta, 0) + q_{22}(\eta, 0)] \\
&+ \frac{\alpha}{\varepsilon}\tilde{e}_1^T M^g \phi_\xi \theta + \frac{\tilde{e}_2^T}{\varepsilon} B^g \phi_\xi \theta \\
&\leq -\frac{\alpha}{\varepsilon}\lambda_{\min}\|\tilde{e}_1\|^2 + \frac{1}{\varepsilon}\left(\alpha \|M^g A_c B\| + \|B^g A_c M\|\right) \|\tilde{e}_1\| \|\tilde{e}_2\| \\
&+ \frac{1}{\varepsilon}\|\tilde{e}_2\|^2 \|B^g A_c B\| + \beta_2 \|\tilde{e}_2\|^2 \varepsilon^{r-1} - \|\tilde{e}_2\|^2 \varepsilon^{r-1}(1+m\beta_1) \\
&+ \frac{\alpha}{\varepsilon}\|\tilde{e}_1\| \|M^g \phi_\xi\| \|\theta\| + \frac{1}{\varepsilon}\|\tilde{e}_2\| \|B^g \phi_\xi\| \|\theta\| + \mu \|\frac{\partial V_0}{\partial \eta}\| \|q_{22}(\eta, \tilde{e}_1) - q_{22}(\eta, 0)\| \\
&+ \mu \left[\left(\frac{\partial V_0}{\partial \eta}\right)^T q_{22}(\eta, 0)\right] + \mu \|\frac{\partial V_0}{\partial \eta}\| \|\phi_n\| \|\theta\| \\
&\leq -\frac{\alpha}{\varepsilon}\lambda_{\min}\|\tilde{e}_1\|^2 + \left(\frac{\alpha}{\varepsilon}\|M^g A_c B\| + \frac{1}{\varepsilon}\|B^g A_c M\|\right)^2 \|\tilde{e}_2\|^2 + \frac{1}{4}\|\tilde{e}_1\|^2 \\
&+ \frac{1}{\varepsilon}\|\tilde{e}_2\|^2 \|B^g A_c B\| + \beta_2 \varepsilon^{r-1} \|\tilde{e}_2\|^2 - \|\tilde{e}_2\|^2 \varepsilon^{r-1}(1+m\beta_1) \\
&+ \frac{\alpha}{\varepsilon}\|\tilde{e}_1\| \|M^g \phi_\xi\| \|\theta\| + \frac{1}{\varepsilon}\|\tilde{e}_2\| \|B^g \phi_\xi\| \|\theta\| \\
&+ \mu L k_4 \|\eta\| \left(\|\tilde{e}_1\| + \|\tilde{e}_2\|\right) - \mu k_3 \|\eta\|^2 + \mu k_4 \|\eta\| \|\phi_n\| \|\theta\| \\
&\leq \|\tilde{e}_1\|^2 \left(-\frac{\alpha}{\varepsilon}\lambda_{\min} + \frac{1}{4} + \left(\frac{\alpha}{\varepsilon}\|M^g \phi_\xi\|\right)^2\right) \\
&+ \|\tilde{e}_2\|^2 \Bigg[\left(\frac{\alpha}{\varepsilon}\|M^g A_c B\| + \frac{1}{\varepsilon}\|B^g A_c M\|\right)^2 + \frac{1}{\varepsilon}\|B^g A_c B\| - \varepsilon^{r-1}(1+m\beta_1) \\
&+ \left(\frac{1}{\varepsilon}\|B^g \phi_\xi\|\right)^2 + \beta_2 \varepsilon^{r-1}\Bigg] \\
&+ \mu L K_4 \|\eta\| \left(\|\tilde{e}_1\| + \|\tilde{e}_2\|\right) - \mu k_3 \|\eta\|^2 + \left(\mu k_4 \|\phi_\eta\|\right)^2 \|\eta\|^2 + \frac{3}{4}\|\theta\|^2
\end{aligned}
$$

$$
\begin{aligned}
&= \|\tilde{e}_1\|^2 \left(-\frac{\alpha}{\varepsilon}\lambda_{\min} + \frac{1}{4} + \left(\frac{\alpha}{\varepsilon}\|M^g\phi_\xi\|\right)^2\right)\\
&+\|\tilde{e}_2\|^2\Bigg[\left(\frac{\alpha}{\varepsilon}\|M^g A_c B\| + \frac{1}{\varepsilon}\|B^g A_c M\|\right)^2 + \frac{1}{\varepsilon}\|B^g A_c B\| - \varepsilon^{r-1}(1+m\beta_1)\\
&+\left(\frac{1}{\varepsilon}\|B^g\phi_\xi\|\right)^2 + \beta_2\varepsilon^{r-1}\Bigg]\\
&-\mu k_3\|\eta\|^2 + \mu L k_4\|\eta\|\|\tilde{e}_1\| + \mu L k_4\|\eta\|\|\tilde{e}_2\| + (\mu k_4\|\phi_\eta\|)^2\|\eta\|^2 + \frac{3}{4}\|\theta\|^2\\
&\le -\|\tilde{e}_1\|^2\left(\frac{\alpha}{\varepsilon}\lambda_{\min} + \frac{1}{4} + \left(\frac{\alpha}{\varepsilon}\|M^g\phi_\xi\|\right)^2\right)\\
&+\|\tilde{e}_2\|^2\Bigg[\left(-\frac{\alpha}{\varepsilon}\|M^g A_c B\| + \frac{1}{\varepsilon}\|B^g A_c M\|\right)^2 + \frac{1}{\varepsilon}\|B^g A_c B\| - \varepsilon^{r-1}(1+m\beta_1)\\
&+\left(\frac{1}{\varepsilon}\|B^g\phi_\xi\|\right)^2 + \beta_2\varepsilon^{r-1}\Bigg] - \mu k_3\|\eta\|^2 + (\mu L k_4)^2\|\eta\|^2 + \frac{1}{4}\|\tilde{e}_1\|^2\\
&+4(\mu L k_4)^2\|\eta\|^2 + \frac{1}{16}\|\tilde{e}_2\|^2 + (\mu k_4\|\phi_\eta\|)^2\|\eta\|^2 + \frac{3}{4}\|\theta\|^2\\
&\le -\|\tilde{e}_1\|^2\left(\frac{\alpha}{\varepsilon}\lambda_{\min} - \frac{1}{2} - \left(\frac{\alpha}{\varepsilon}\|M^g\phi_\xi\|\right)^2\right)\\
&-\|\tilde{e}_2\|^2\Bigg[\varepsilon^{r-1}(1+m\beta_1) - \left(\frac{\alpha}{\varepsilon}\|M^g A_c B\| + \frac{1}{\varepsilon}\|B^g A_c M\|\right)^2 - \frac{1}{\varepsilon}\|B^g A_c B\|\\
&-\frac{1}{16} - \left(\frac{1}{\varepsilon}\|B^g\phi_\xi\|\right)^2 - \beta_2\varepsilon^{r-1}\Bigg]\\
&-\|\eta\|^2\left[\mu k_3 - 5(\mu L k_4)^2 - (\mu k_4\|\phi_\eta\|)^2\right] + \frac{3}{4}\|\theta\|^2\\
&\le -N_2\left(\|\tilde{e}_1\|^2 + \|\tilde{e}_2\|^2 + \|\eta\|^2\right) + \frac{3}{4}\|\theta\|^2 := -N_2\|y_{total}\|^2 + \frac{3}{4}\|\theta\|^2 \qquad (2.48)
\end{aligned}
$$

where

$$\|y_{total}\|^2 := \|\tilde{e}_1\|^2 + \|\tilde{e}_2\|^2 + \|\eta\|^2. \qquad (2.49)$$

By virtue of (Khalil, 1996, Theorem 5.2), (2.48) implies the input-to-state stability for the closed-loop system. Furthermore, it is easy to see that

$$\omega_1\left(\|\tilde{e}_1\|^2 + \|\tilde{e}_2\|^2 + \|\eta\|^2\right) \le V \le \omega_2\left(\|\tilde{e}_1\|^2 + \|\tilde{e}_2\|^2 + \|\eta\|^2\right)$$

i.e.

$$\omega_1\|y_{total}\|^2 \le V \le \omega_2\|y_{total}\|^2 \qquad (2.50)$$

where $\omega_1 := \min\left(\frac{\alpha}{2}, \frac{1}{2}, \mu k_1\right)$, $\omega_2 := \max\left(\frac{\alpha}{2}, \frac{1}{2}, \mu k_2\right)$. From (2.48) and (2.50), we get

$$\dot{V} \leq -\frac{N_2}{\omega_2}V + \frac{3}{4}\|\theta\|^2 \leq -\frac{N_2}{\omega_2}V + \frac{3}{4}\left(\sup_{t_0\leq\tau\leq t}\|\theta(\tau)\|\right)^2 \tag{2.51}$$

Hence,

$$V(t) \leq V(t_0)e^{-\frac{N_2}{\omega_2}(t-t_0)} + \frac{3\omega_2}{2N_2}\left(\sup_{t_0\leq\tau\leq t}\|\theta(\tau)\|\right)^2 \tag{2.52}$$

which implies

$$|e_1(t)| \leq \sqrt{V(t_0)}e^{-\frac{N_2}{2\omega_2}(t-t_0)} + \sqrt{\frac{3\omega_2}{2N_2}}\left(\sup_{t_0\leq\tau\leq t}\|\theta(\tau)\|\right). \tag{2.53}$$

Hence, statement (2.23b) is proved. From (2.19c) and (2.48), we get

$$\dot{V} \leq -N_2\left(\|\tilde{e}_1\|^2 + \|\tilde{e}_2\|^2 + \|\eta\|^2\right) + \frac{3}{4}\|\theta\|^2 \tag{2.54a}$$

i.e.

$$\dot{V} + N_2\|e_1\|^2 \leq -N_2\left(\|\tilde{e}_2\|^2 + \|\eta\|^2\right) + \frac{3}{4}\|\theta\|^2 \tag{2.54b}$$

which implies

$$\int_{t_0}^{t}(y(\tau) - y_d(\tau))^2\,d\tau \leq \frac{V(t_0)}{N_2} + \frac{3}{4N_2}\int_{t_0}^{t}\|\theta(\tau)\|^2\,d\tau \tag{2.55}$$

so that statement (2.23c) is satisfied. Finally, we will prove that the sphere $B_{\underline{r}}$ is a global attractor for the output tracking error of system (2.1). From (2.48) and (2.26g), we get

$$\dot{V} \leq -N_2\left(\|y_{total}\|^2\right) + N_1 \tag{2.56}$$

For $\|y_{total}\| \geq \underline{r}$, we have $\dot{V} < 0$. Hence any sphere defined by

$$B_{\underline{r}} := \left\{\begin{bmatrix}\bar{e}\\ \eta\end{bmatrix} : \|\bar{e}\|^2 + \|\eta\|^2 \leq \underline{r}\right\} \tag{2.57}$$

is a global final attractor for the tracking error system of the nonlinear control systems (2.1). Furthermore, it is easy routine to see that, for $y_{total} \notin B_{\underline{r}}$, we have

$$\begin{aligned}\frac{\dot{V}}{V} &\leq \frac{-N_2\|y_{total}\|^2 + N_1}{V} \leq \frac{-N_2\|y_{total}\|^2 + N_1}{\omega_2\|y_{total}\|^2}\\ &\leq \frac{-N_2}{\omega_2} + \frac{N_1}{\omega_2\|y_{total}\|^2} \leq \frac{-N_2}{\omega_2} + \frac{N_1}{\omega_2\underline{r}^2} := -\alpha^*\end{aligned} \tag{2.58}$$

i.e.,

$$\dot{V} \le -\alpha^* V$$

According to the comparison theorem (Miller and Michel, 1982), we get

$$V(y_{total}(t)) \le V(y_{total}(t_0)) \exp\left[-\alpha^*(t - t_0)\right]$$

Therefore,

$$\begin{aligned} &\omega_1 \left\|y_{total}\right\|^2 \le V(y_{total}(t)) \le V(y_{total}(t_0)) \exp\left[-\alpha^*(t - t_0)\right] \\ &\le \omega_2 \left\|y_{total}(t_0)\right\|^2 \exp\left[-\alpha^*(t - t_0)\right] \end{aligned} \tag{2.59}$$

Consequently, we get

$$\left\|y_{total}\right\| \le \sqrt{\frac{\omega_2}{\omega_1}} \left\|y_{total}(t_0)\right\| \exp\left[-\frac{1}{2}\alpha^*(t - t_0)\right]$$

i.e., the convergence rate toward the sphere $B_{\underline{r}}$ is equal to $\alpha^*/2$. This completes our proof. ■

If the relative degree of nonlinear control system is equal to one, then Theorem 1 will be reduced to the simplified version as follows:

Assumption 3 *For all $t \ge 0$, $\eta \in \Re^{n-r}$ and $\xi \in \Re^r$, there exists a positive constant L such that the following inequality holds:*

$$\left\|q_{22}(\eta, \tilde{e}) - q_{22}(\eta, 0)\right\| \le L\left(\left\|\tilde{e}_2\right\|\right) \tag{2.60}$$

where $q_{22}(\eta, \tilde{e}) := q(\xi, \eta)$.

Assumption 4 *There exists known functions $\beta_1(\cdot), \beta_2(\cdot) : \Re^n \to R^+$ such that*

$$1 + \delta_2(X) \ge \beta_1 \tag{2.61a}$$

$$\left\|\delta_2 y_d^{(1)} - (1 + 2\delta_2)\varepsilon^{-1}\bar{\bar{e}} + \delta_1 d - \delta_2 c + \bar{e}\right\| \le \beta_2 \left\|\tilde{e}_2\right\| \tag{2.61b}$$

where

$$\bar{\bar{e}} = \alpha_1 \bar{e}_1. \tag{2.62}$$

Theorem 2 *Suppose that there exists a continuously differentiable function $V_0 : \Re^{n-r} \to \Re^+$ such that the following three inequalities hold for all $\eta \in \Re^{n-r}$:*

$$\text{(a)} \quad k_1\|\eta\|^2 \le V_0(\eta) \le k_2\|\eta\|^2, \quad k_1, k_2 > 0 \tag{2.63}$$

$$\text{(b)} \quad (\nabla_\eta V_0)^T q_{22}(\eta, 0) \le -k_3\|\eta\|^2, \quad k_3 > 0 \tag{2.64}$$

$$\text{(c)} \quad \|\nabla_\eta V_0\| \le k_4\|\eta\|, \quad k_4 > 0, \tag{2.65}$$

then the tracking problem with almost disturbance decoupling is globally solvable by the controller defined by

$$u_{feedback} = [L_g h(X(t))]^{-1} \left\{ -L_f^1 h(X) + y_d^{(1)} - 2\varepsilon^1 \alpha_1 [L_f^0 h(X) - y_d] - m\bar{e} \right\} \quad (2.66)$$

where m is adjustable positive constant. Moreover, the influence of disturbances on the L_2 norm of the tracking error can be arbitrarily attenuated by increasing the adjustable parameter $N_2 > 1$:

$$N_2 = \min\{k_{22}, k_{33}\} \quad (2.67)$$

$$k_{22} := -\frac{1}{\varepsilon}\|B^g A_c B\| + m\beta_1 + \frac{15}{16} - \left(\frac{1}{\varepsilon}\|B^g \phi_\xi\|\right)^2 - \beta_2 \quad (2.68)$$

$$k_{33} := \mu k_3 - 5(\mu L k_4)^2 - (\mu L k_4 \|\phi_\eta\|)^2 \quad (2.69)$$

$$N_1 := \frac{3}{4}\left(\sup_{t_0 \le \tau \le t} \|\theta(\tau)\|\right)^2. \quad (2.70)$$

2.2. Fuzzy controller design

After using feedback linearization control as a guarantee of uniform ultimate bounded stability, the multiple input/single output fuzzy control design can be technically applied via human expert's knowledge to improve the convergence rate of tracking error dynamics. The block diagram of the fuzzy control is shown in Fig. 1. In general, the tracking error $e(t)$ and its time derivative $\dot{e}(t)$ are utilized as the input fuzzy variables of the IF-THEN control rules and the output is the control variable u_{fuzzy}.

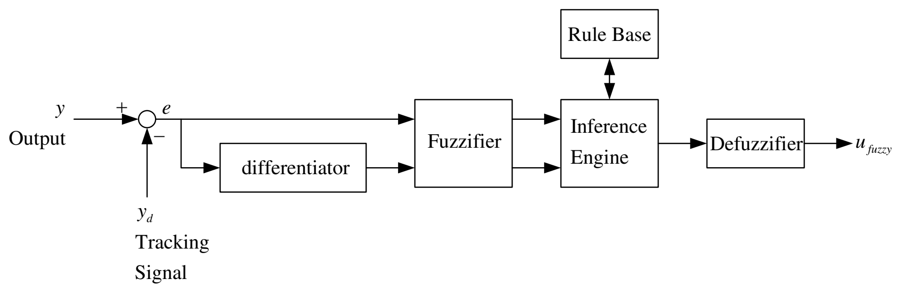

Figure 1. Fuzzy logic controller

For the sake of easy computation, the membership functions of the linguistic terms for $e(t)$, $\dot{e}(t)$ and u_{fuzzy} are all chosen to be the triangular shape function. We define seven linguistic terms: PB (Positive big), PM (Positive medium), PS (Positive small), ZE (Zero), NS (Negative small), NM (Negative medium) and NB (Negative big), for each fuzzy variable, as shown in Fig. 2.

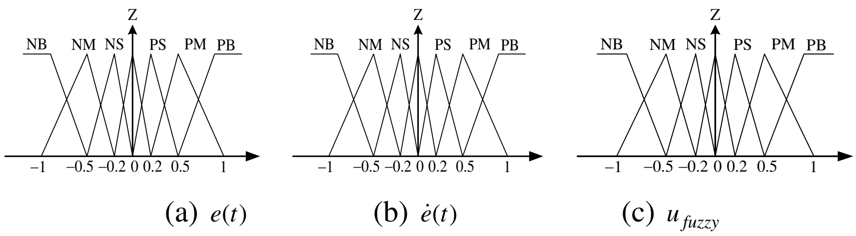

Figure 2. Membership functions for (a) $e(t)$ (b) $e(t)$ and (c) u_{fuzzy}

Fuzzy control rule table for u_{fuzzy} is shown in Table 1. The rule base is heuristically built by the standard McVicar-Whelan rule base (Yager and Filev, 1994) for usual servo control systems. The Mamdani method is used for fuzzy inference. The defuzzification of the output set membership value is obtained by the centroid method. Therefore, we can combine the designs of feedback linearization control and fuzzy control to construct the overall controller as follows:

$$\begin{aligned}
&u_{fe+fu} := u_{feedback}u_s(t) + u_{fuzzy}u_s(t-t_1)\\
&= [L_g L_f^{r-1} h(X(t))]^{-1}\Big\{ - L_f^r h(X) + y_d^{(r)} - 2\varepsilon^{-r}\alpha_1[L_f^0 h(X) - y_d]\\
&\quad -2\varepsilon^{1-r}\alpha_2[L_f^1 h(X) - y_d^{(1)}] - \dots\\
&\quad -2\varepsilon^{-1}\alpha_r[L_f^{r-1}h(X) - y_d^{(r-1)}] + mB^g\bar{e}\Big\}u_s(t) + u_{fuzzy}u_s(t-t_1) \qquad (2.71)
\end{aligned}$$

where u_s denotes the unit step function and t_1 is the time when the tracking error dynamics of the system touch the final attractor $B_{\underline{r}}$.

Table 1. Fuzzy control rule base

u_{fuzzy}		$e(t)$						
		NB	NM	NS	ZE	PS	PM	PB
$\dot{e}(t)$	NB	PB	PB	PB	PB	PM	PS	ZE
	NM	PB	PB	PB	PM	PS	ZE	NS
	NS	PB	PB	PM	PS	ZE	NS	NM
	ZE	PB	PM	PS	ZE	NS	NM	NB
	PS	PM	PS	ZE	NS	NM	NB	NB
	PM	PS	ZE	NS	NM	NB	NB	NB
	PB	ZE	NS	NM	NB	NB	NB	NB

3. Illustrative example

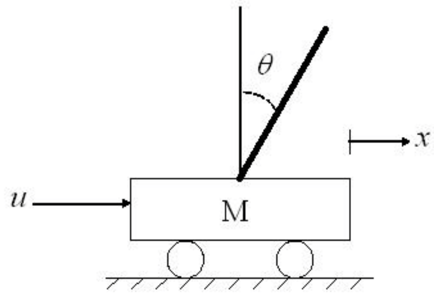

Figure 3. Inverted pendulum control system (θ: angle of pendulum)

The inverted pendulum on a cart, shown in Fig. 3, is a famous unstable highly nonlinear system. The dynamic behavior of the inverted pendulum system is described by four state variables: x=position variable of the cart on the track, $\dot{x}$=velocity variable of the cart, θ=angle variable of the pendulum and $\dot{\theta}$=angular velocity variable of the pendulum. Assume that the pendulum is freely hinged to the cart, which is free to move on a horizontal track with no moment of inertia and viscous friction for the motion of the cart. Then the dynamic equations of motion for the inverted pendulum system can be derived as follows based on the second motion law of Newton:

$$(M+m)\ddot{x} + ml\ddot{\theta}\cos\theta - ml(\dot{\theta})^2\sin\theta = u \tag{3.1a}$$

$$ml^2\dot{\theta} + ml\cos\theta\ddot{x} = mgl\sin\theta \tag{3.1b}$$

where u is an input force, M is the mass of the cart, m is the mass of the pendulum, $2l$ is the length of the pendulum and g is the gravitational constant. Due to the thickness of the pendulum, the angle is not greater or equal to $\pi/2$ (i.e., $\theta < \pi/2$). With the choices of the state variables and the output $x_1 = x$, $x_2 = \dot{x}$, $x_3 = \theta$, $x_4 = \dot{\theta}$, $y = \theta + \dot{\theta}$, the dynamic equations of the inverted pendulum system are

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} x_2 \\ \dfrac{-mgl\sin x_3\cos x_3 + ml^2(\sin x_3)\,x_4^2}{l(M+m) - ml\cos^2 x_3} \\ x_4 \\ \dfrac{g(M+m)(\sin x_3) - ml\sin x_3\left(x_4^2\right)}{l(M+m) - ml\cos^2 x_3} \end{bmatrix} +$$

$$+\begin{bmatrix} 0 \\ \dfrac{l}{l(M+m)-ml\cos^2 x_3} \\ 0 \\ \dfrac{-\cos x_3}{l(M+m)-ml\cos^2 x_3} \end{bmatrix} u + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \theta \tag{3.2a}$$

$$y = x_3 + x_4 \tag{3.2b}$$

where the noises are assumed to be $\theta = \sin(t-2)$.

In the simulation, the true values of the parameters are $M = 2kg$, $m = 0.2kg$, $l = 0.6m$ and $g = 9.8m/s^2$. Then the dynamics equations can be rewritten as follows:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} x_2 \\ \dfrac{-1.176\sin x_3 \cos x_3 + 0.072\,(\sin x_3)\, x_4^2}{1.32 - 0.12\cos^2 x_3} \\ x_4 \\ \frac{21.56\sin x_3 - 0.12\sin x_3 \left(x_4^2\right)}{1.32-0.12\cos^2 x_3} \end{bmatrix}$$

$$+\begin{bmatrix} 0 \\ \frac{0.6}{1.32-0.12\cos^2 x_3} \\ 0 \\ \dfrac{-\cos x_3}{1.32-0.12\cos^2 x_3} \end{bmatrix} u + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \theta \tag{3.3a}$$

$$y = x_3 + x_4 = h(X) \tag{3.3b}$$

We will consider exponential tracking of the output $y(t) = h(X(t)) = x_3 + x_4$ to a desired signal $y_d(t) = 0$ (the pendulum is located at vertical position, i.e.,$x_3 = 0$). The original system (3.3) is a system of relative degree one. It can be verified that with the choice $\varepsilon = 1$, $V_0(\eta) = \eta_2^2 + \eta_3^2 + \eta_4^2$, $\mu = 1.67$, $m = 2$ and $\alpha_1 = 1$, the related conditions of Theorem 2 are satisfied with $\xi_1 = x_3 + x_4$, $\eta_2 = 0.1x_3$, $\eta_3 = 0.1x_3$, $\eta_4 = 0.1x_3$, $r = 1$, $B = B^g = 1$, $L = \sqrt{0.03}$, $A_c = -1$, $k_1 = k_2 = 1$, $k_3 = k_4 = 2$, $\|\phi_\zeta\| = \|\phi_\eta\| = 0$, $k_{22} = 8.75$, $k_{33} = 8.34$, $N_1 = 0.75$, $N_2 = 8.34$, $\beta_1 = 1$ and $\beta_2 = 0$. The desired tracking controller is given below

$$\begin{aligned} u_{fe+fu} = {} & \left(\frac{-\cos x_3}{1.32 - 0.12\cos^2 x_3}\right)^{-1} \Bigg(-\left(x_4 + \frac{21.56\sin x_3 - 0.12\sin x_3 (x_4^2)}{1.32 - 0.12\cos^2 x_3} \right) \\ & -(2\varepsilon^{-1}\alpha_1 + 40)(x_3 + x_4) \Bigg) u_s(t) + u_{fuzzy} u_s(t - t_1) \end{aligned} \tag{3.4}$$

The tracking errors driven by u_{fe+fu} for (3.1) is depicted in Fig. 4.

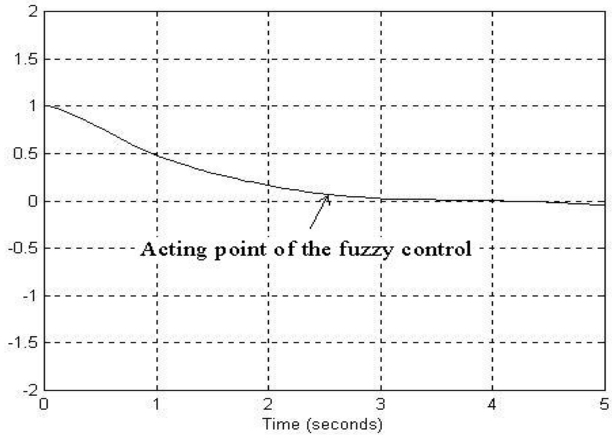

Figure 4. The tracking error driven by u_{fe+fu} for (3.1)

4. Comparative example with an existing approach

It is shown in Marino and Tomei (1999) for the following example

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} \theta_1(t) \\ x_2^3\theta_2(t) \end{bmatrix} \tag{4.1a}$$

$$y(t) = x_1(t) := h(X(t)) \tag{4.1b}$$

that the tracking and almost disturbance decoupling problem cannot be solved, with $\theta_1 = \theta_2 = 0.5\sin t$. The fuzzy feedback linearization control algorithm proposed in this paper will solve it perfectly. Applying the same design procedures of Theorem 1 yields the desired tracking and almost disturbance decoupling controller as follows:

$$u_{fe+fu} = \left(1 + x_2^2\right)\ [-\sin t - 40(x_1 - \sin t + \tan^{-1} x_2 - \cos t)]u_s(t) + u_{fuzzy}u_s(t-t_1) \tag{4.2}$$

The tracking error dynamics driven by u_{fe+fu} for (4.1) is depicted in Fig. 5.

5. Conclusion

In this paper we have constructed a fuzzy feedback control algorithm which globally solves the tracking problem with almost disturbance decoupling. The discussion and practical application of input-output feedback linearization of nonlinear control systems with uncertainties by parameterized co-ordinate transformation have been presented. One comparative example is proposed to show the

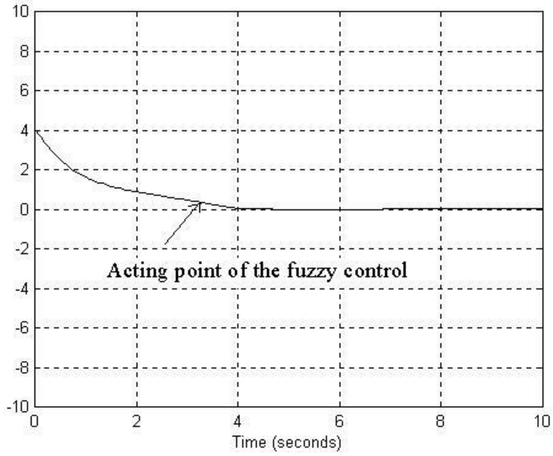

Figure 5. The tracking error driven by u_{fe+fu} for (4.1)

significant contribution of this paper with respect to some existing approaches. Moreover, a practical example of an inverted pendulum control system demonstrated the applicability of the proposed differential geometry approach and the composite Lyapunov approach. Simulation results exploited the fact that the proposed methodology is successfully applied to input-output linearization problem and achieves the desired tracking and almost disturbance decoupling performances of the controlled system.

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